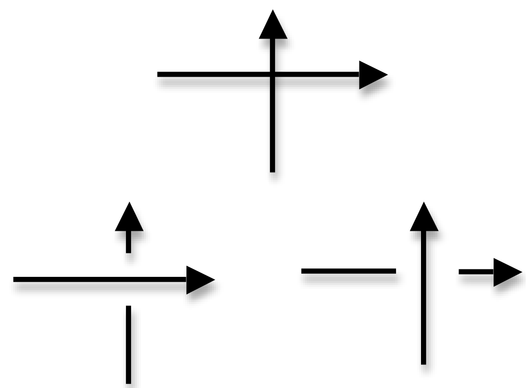
Notes From Tait's *On Knots* To Lacan's *Sinthome* – Course-Seminar on Topology, Logic, and Analysis 2001-2011

Robert Groome, PLACE, Santa Monica, CA
Res1d6qq@verizon.net

0. Crossing Number

Since Gauss [ ], Listing [ ], and Tait[ ], the classical theory of knots takes the position that the *planar graph* is a *projection* of the knot and then calls a *diagram* a projection in which the *overs* and *unders* have been established at each crossing. As a consequence, there are two choices at each crossing either labeled +1 or –1. What is interpreted in the diagram as '*over*' is the '*height*' of a string that is represented in the diagram by a broken trait.

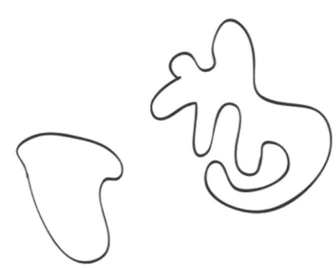


+1 **–1**

One of the oldest knot invariants in the history of knot theory is the *crossing number* stating:

0.1. For any knot there is a minimal number of crossings.

But to find this minimal number we first have to find a way of determining the trivial crossings – or tangles – of a presentation. The suspicion is that there are an infinite number of diagrams with a different amount of crossings that *represent* one and the same knot, but only a finite number of diagrams that have a *minimal amount* of crossings. The trivial knot – or closed curve – for example, has an infinity of diagrams with a different amount of crossings, but only one diagram that is the same as all the others with a minimum of crossings, e.g., zero. The minimal amount of crossings of a diagram has, since Tait, been called the *crossing number* **C(k)** and is an invariant of classical knot theory. It is this strategy of reading the knot that has been inherited today: enumerate all the possible diagrams of a knot, then group those together that *represent* the same object in space as a *knot-type* (a class of equivalent diagrams).



Reduced Diagram

C(k) = 0

1. The Classical Tait-Conjectures

By seeking to determine the conditions for determining minimal presentation of the crossings of a knot, Tait began to determine not only how to go from a knot-diagram to the knot, but to delimit a field of conjectures that have retrospectively been called the *Classic Tait Conjectures.* We call these conjectures here, '*Classic*' for the reason that they have been retrospectively isolated by the consensus of a post-Tait mathematical community. As we will show later, these *Classic Conjectures* neither exhaust nor go to the heart of Tait's theory of knots that is much more focused around the problem of *Locking* or *Borromeaness*. Before turning to this, let us turn to the problem of finding a method to reduce a knot to its *minimal presentation* of crossings.

First, if the *crossing number* **C(k)** is an *invariant*, then it is an invariant 'of': which means, it names an obstacle to a movement – an *isotopy* – in the plane.

One type of obstacle is *alternation*: in traversing a knot in a particular direction that you can pass over, then under – or vice versa – around the whole knot. This is not an invariant of the knot because there are non-alternating knots.

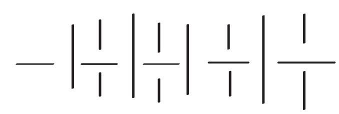
Another type of obstacle is *minimal alternation*: if a knot is *alternable*, then it can be shown that it has minimal amount of crossings for any projection of that knot (*Conjecture #1*) and that any two reduced alternating diagrams of the same knot have the same amount of crossings (*#2*). The proof of the *Classic Tait Conjectures* has shown that the *crossing number* is an *invariant* for the theory of knots.

Second, the *Classic Conjectures* suppose that one is working with only *alternating knots;* what is assumed is that a knot K is *alternable*, i.e., that it can be put into an *alternating projection*. In fact, the problem of *non-alternable* knots only begins with the cases where **C(k)=8**; while their existence was first noted by Tait [ ] , then enumerated and classified by Little [ ].

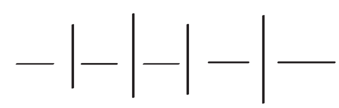
Classical
Tait Conjectures:

1 *One alternating projection of a knot is reduced if it has the least amount of crossings for any projection of that knot.*

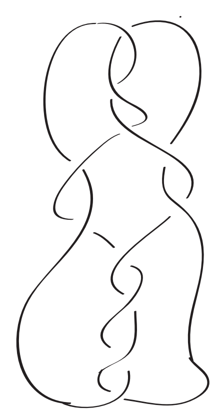
2 *Two reduced alternating projections of the same knot have the same amount of crossings.*



Alternating Crossings



Non-Alternate Crossings



Alternable Knot with non-minimal presentation.

Leaving aside the case of *non-alternable* knots for the moment, the problem of determining the crossing number becomes one of finding a way to reduce a projection of the knot by sliding – *isotopy* – to a minimal presentation or '*reduced diagram*'. It should be noticed that if K is an alternating knot, this does not mean that one can simply subtract the tangling to reduce it to a minimum. Indeed, Goeritz [1934] showed that there are knot-diagrams that require adding more tangling before they can become untangled into a *reduced diagram*.

Alternable			**Non-Alternable**	
Minimal	**Nonminmal**		**Minimal**	**Nonminimal**
	Alternate	**Nonalternate**