3 Introduction To Lock Theory

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The following is a pre-print of Chpt. 3 of the book: Return To P.G. Tait via Analytic Theory Of Knots. I introduce the curious problem of 'Borromeaness', or Locking as it was first coinded by P.G. Tait, then show the difficulties it causes in the calculation of the linking number. The questions are raised: What is the difference between a link and non-link? Are locks and trivial links the only types of non-links? What is the movement that establishes a link as invariant? Classify this movement into three types (the Reidemeister Moves) and develop a calculation of both these movements and what poses an obstacle to these movements (an invariant). The text has as its goal to transmit to those working in the field of analysis the necessary material to construct the topological problems posed by the late J. Lacan without reducing such constructs to the repetition of commentary. *This is a pre-print article for the internet.*

By Robert Grome

0- An informal introduction to Links and Non-Links

Though we assume no mathematical background here, a previous lecture of *Chapters §1* and *§2* will facilitate a reading of the psychoanalytic implications.

1- Crossings In Immersed Graphs and Embedded Knots

First, we will place ourselves within the framework of a classical formal theory as explained in *Section §1*. Within this framework we will suppose that we are working with the projection of an object in three-dimensional space on a two-dimensional surface. As explained in the last section, a knot projection results in an immersed graph having crossings points at the meeting of two strands of the projected 3-d embedded knot or link:

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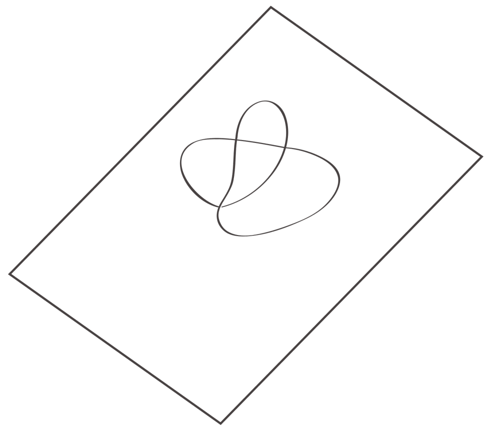
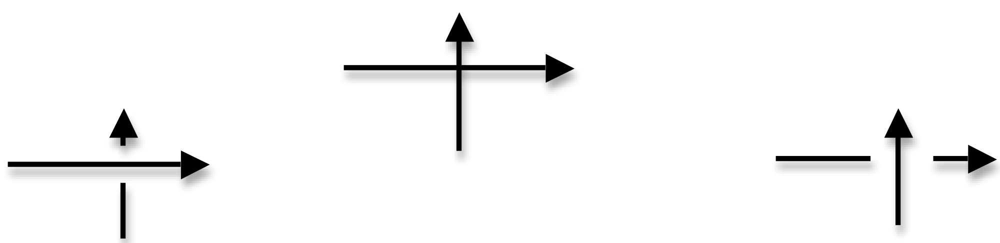
[1] -Coat of arms showing Borromean interlocking

Embedded Knot in 3-d

Immersed Graph in 2d



As demonstrated, within the classical framework, for every **n** crossing points in the plane, there are $\mathbf{2^n}$ possible over and under diagrams in space:



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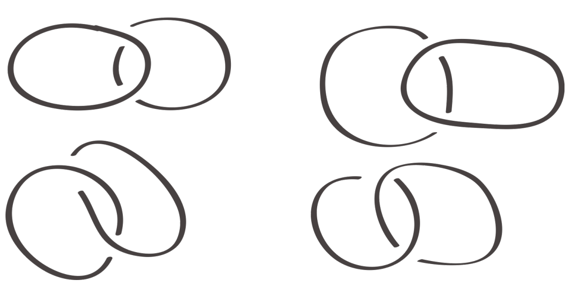
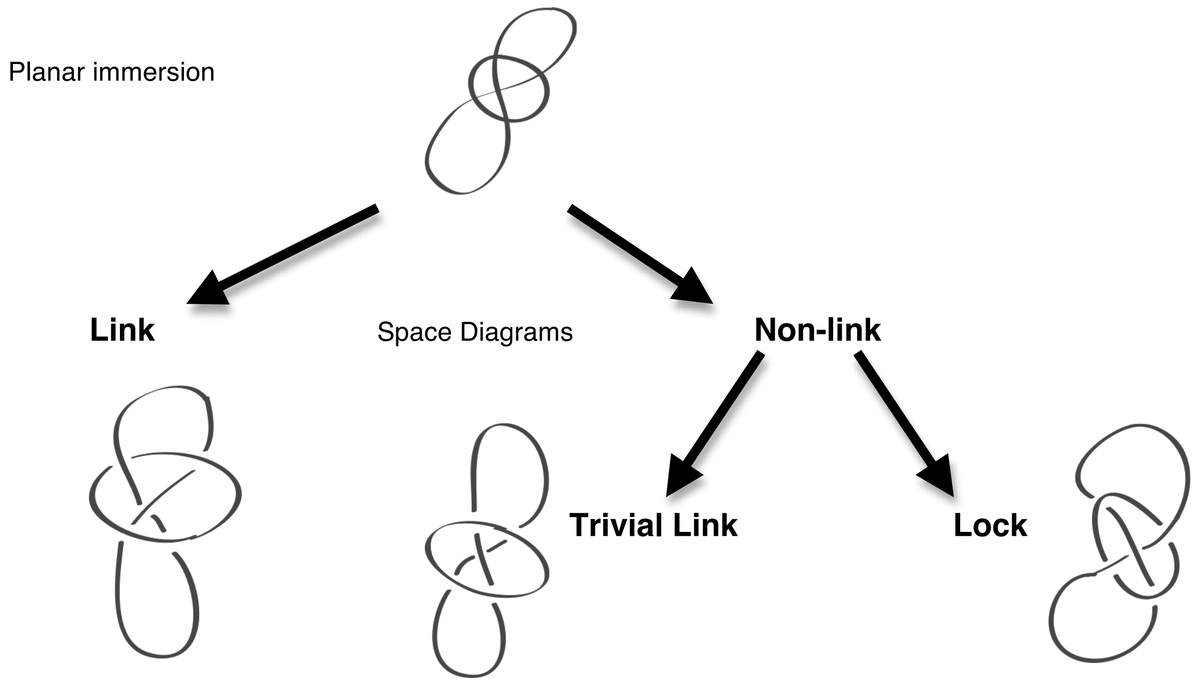
For example, given an immersed graph of two intersecting closed curves at **n** points, there are $\mathbf{2^2 = 4}$ different spatial embeddings:



In this section, we will only be interested in multiple component *locks* and *links*, leaving until a later the problem of single component knots.

2- Linking, Locking, Trivial Link, or Something Else?

It is important to recognize that the choice of $\mathbf{2^n}$ *overs* and *unders* on an immersed graph determines whether the diagram is *linked*, *locked*, or *trivial*, i.e., an unconnected collections of closed curves. Schematically, this choice of *overs* and *unders* results in a type of diagram that can be presented in a decision tree:



Informally, let us fix our vocabulary with the following definitions:

1) **(X,Y)** is a **trivial link** if and only if the closed curve **X** does not borrow the hole of the other closed curve **Y** and the two closed curves are *separable*, i.e., it is possible to completely disconnect one component from the other by sliding in the plane (isotopy)

2) **(X,Y)** is a **link** if and only if the closed curve **X** does borrow the hole of the other closed curve **Y** and the two closed curves are <u>not</u> *separable*, i.e., it is impossible to completely disconnect one component from the other by sliding in the plane (isotopy).

3) **(X,Y)** is a **lock** if and only if the closed curve **X** does not borrow the hole of the other closed curve **Y** and the two closed curves are <u>not</u> *separable* (see (2) above).

The reader is left as an exercise the generalization of these definitions into more than two components.

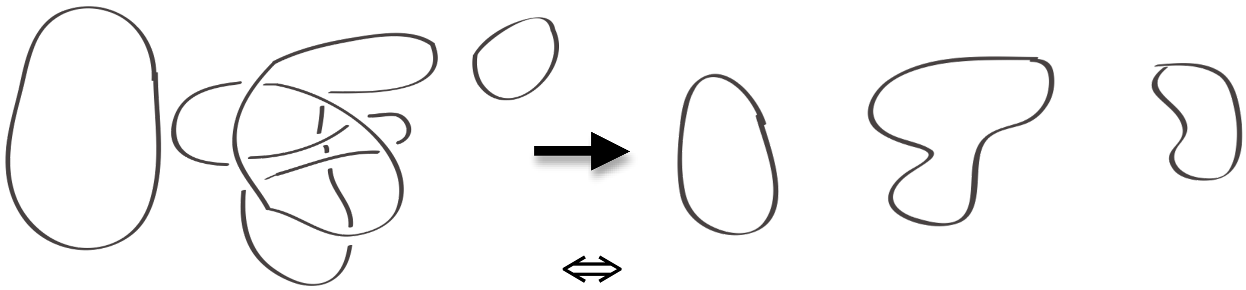
The obvious question is whether these three types of connection between closed curves are exhaustive and the degree of combination and complexity that there may be between them. In any case, given one planar immersion we have to determine whether the resulting embedding diagram is either a *link*, *lock*, *trivial link* or *something else*. The rest of this section aims to introduce different manners of doing so.

3- First Characters of Link Theory: The Reidemeister Moves And Isotopy

In order to respond precisely to the question of whether a diagram is a *link*, *lock*, *trivial link*, we will need a method of writing or calculation to aid us in our decision. In order to introduce this method, I will briefly return to one of the oldest invariants in topology, the *Linking Number*, first discovered by Gauss in [ ]. This will allow us also to introduce a systematic way of accounting for a link by the manner it moves in space via diagrammatic *Reidemeister* moves. As we will show, the weakness of the linking number and the *Reidemeister* moves is that they do not detect a second modality of connection between closed curves first called by Tait *'locking'* [ ].

Given a collection of immersed closed curves in the plane, determine a choice of *overs* and *unders*:

Then determine if the diagram denotes a *link* or *non-link*. If it is a *non-link*, determine if it is a *tangle,* a *lock*, or *something else*. For example, the diagram d.x above denotes a *tangle* because none of the closed curves encounter an obstacle in sliding it across the plane, i.e., all of its components are *separable*. As result of sliding the closed curves, they can be shown to be disconnected:



If the sliding does encounter an obstacle, that is to say, if there is an arc of another closed curve that snags and prevents it from sliding apart, then it is not *separable* and is either a *link* or a *lock*.

For example, the diagram below shows the two closed curves as being connected in the sense that one curve goes inside the other in such a way that the two closed curves snag, .i.e, can not come apart without breaking the other.

In order to tighten up our definitions, let us give a name to sliding in the plane: *isotopy*. Then let us agree to call an obstacle to an *isotopy* an *invariant,* or more precisely, an *invariant of isotopy*. If we assume the arcs are connected to form two closed curves in the above diagram, then such a snag to an *isotopy*, would be called a *link*, i.e., a link is an *invariant of isotopy*.

Let us also fix our vocabulary here by noting that there are three characters – or configurations of traits – that exhaust the possible types of *isotopies* of string diagrams[2]. These *characters of movement* are called the *Reidemeister Moves*:

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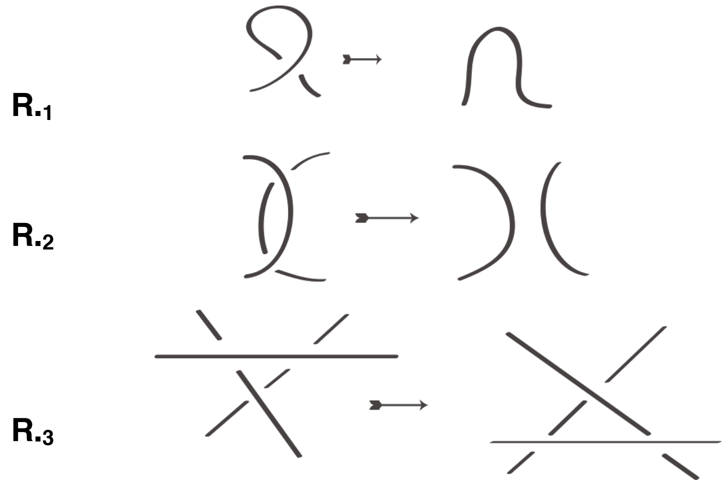
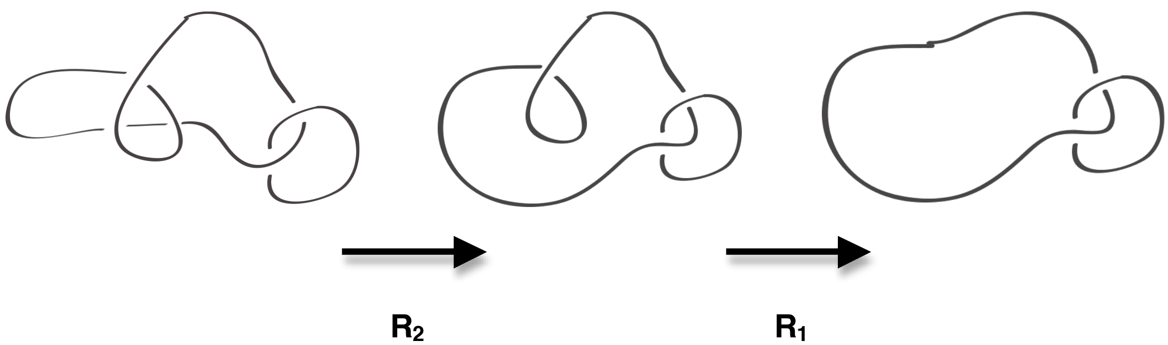
[2] We have noted elsewhere in *Section* §x that these three moves do, however, take for granted the seemingly trivial rotation of a configuration in the plane.



Without becoming too precise at this point of our presentation, it suffices to say here that if there is an obstacle to an *isotopy*, then there is an obstacle to a *Reidemeister* move $\mathbf{R_1}$,$\mathbf{R_2}$, or $\mathbf{R_3}$. For example, in the diagram below, three *Reidemeister* moves are possible, before arriving an obstacle that no *Reidemeister* can undo. Said otherwise, the *Reidemeister* moves remove all the tangling shown in the diagram:



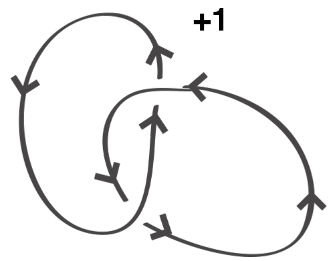
If **X** is a curve in the plane, then a series of Reidemeister $\mathbf{R_x}$, $\mathbf{R_y}$, … $\mathbf{R_n}$ moves performed on **X** may be written as: $\mathbf{R_n}(\mathbf{R_y}(\mathbf{R_x}(X))$. If after a finite number of moves $\mathbf{R_{n-1}}$, the object of the diagram is still obstructed, then we can write an equation that $\mathbf{R_{n-1}}(\mathbf{R_y}(\mathbf{R_x}(X)) \geq \mathbf{1}$, if not **0**, where '**1**' denotes an obstacle and '**0**' does not.

Though this finite list of movements is quite a progress in cataloging obstacles to *isotopy*, it does not allow us to determine whether, given configuration of closed curves, whether the 0-case of non-linking denotes a lock or a trivial link. That is to say, the *Reidemeister* moves by themselves may give us a positive manner to determine linking, but they do not allow for us to determine non-linking.

For example, in an exceedingly complicated diagram, just because we have not arrived to slide things apart according to the *Reidemeister* moves, does not mean we cannot do so; we may have just not arrived at the correct combination. Thus, what is required to overcome this barrier is a method of writing that will allow us to establish which crossings **n** of a multiple of closed curves **(X,Y, …Z_n)** are nugatory without needing to actually slide things around. Such a method of writing is at hand and is called the *Linking Number* **L(k)** where:

$$L(k) = \sum (A_1 + A_2 + \dots A_n)/2.$$

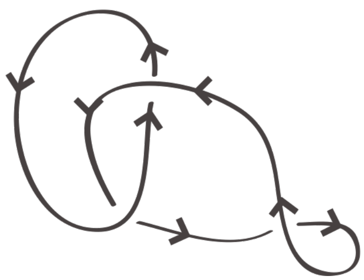
Which is nothing other than to say that the *Linking Number* is the sum of the crossings **A_1, A_2, … A_n** of the diagram divided by *two*. The simplest case of linking occurs in the case where two closed curves share the hole of the other one time. In order to calculate **L(k),** we must first orient the closed curves as follows:



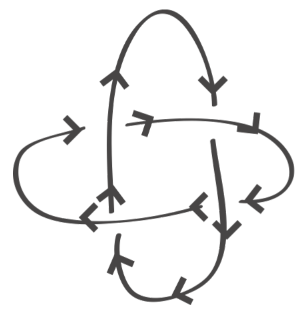
Once this orientation is introduced, the *Linking Number* may be calculated according to the formula: **$L(k) = \sum (A_1 + A_2 + \dots A_n)/2$**. Thus, in accounting for the crossings in the diagrams below we write: **(+1 + (+1))/2 = +1.** Which means the diagram of the object is linked one time in the positive sense of the arrows.

The *Linking Number* allows us to construct an equation– or *decision procedure* – by labeling the diagram in such a way that it allows both the calculation of how many times closed curves are linked and a *decision* as to whether the diagram of an object is actually linked. Said otherwise, if the *Linking Number* **L(k) ≥ 1**, then object is linked one or more times, if the *Linking Number* **L(k) = 0**, then the object is non-linked. But this is again the sticking point: if the object is non-linked, then it is either a *trivial link* – a collection of disconnected closed curves – or it is a *lock*. Leaving until later and examination of whether it may be something else than these three types of connection, I want to focus here on the weakness of the *Linking Number*: it can only detect obstacles that create *linking* and not those that create *locking* as it confuses the latter with the *Unlink* or *Trivial link*.

It is important to recognize that the linking number **L(X)** does not recognize *self-linking* on one strand: that is to say, when you label the configuration do not label those crossings of a strand that crosses itself like the one on the left, only those crossings formed at the intersection of separate components:

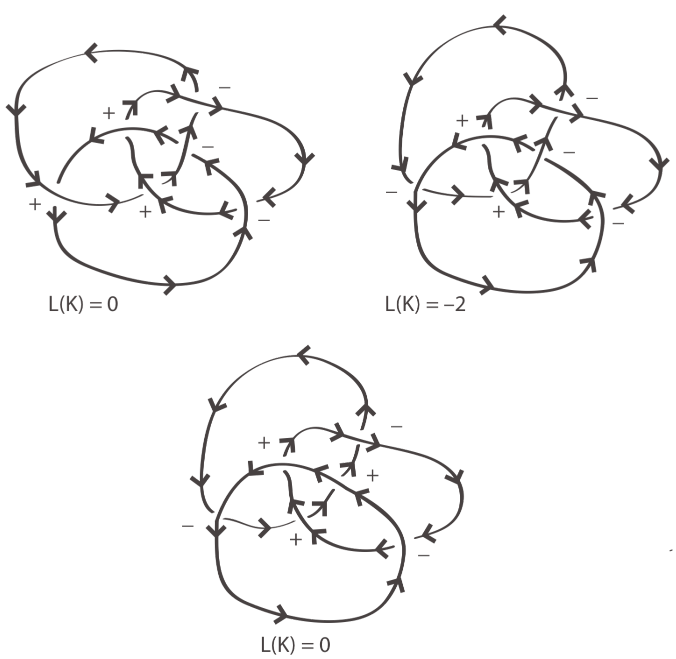


L(k) = ∑(1+1)/2 = 1



L(k) = ∑(1+1+1+1)/2 = 2

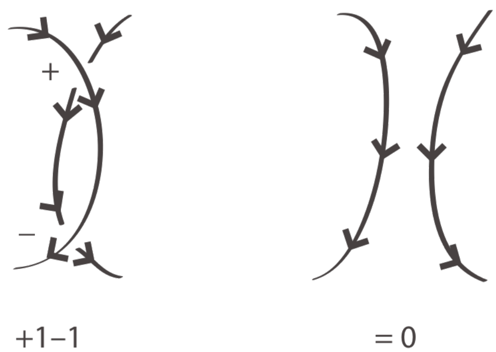
It is also important to recognize how the *Linking Number* computes non-linking as zeros. For example, two of the diagrams below have an **L(k) = 0** and one an **L(K) = –2** :



But only the lower configuration of closed curves is separable into a trivial link. The other has a mode of connection undetected by the *Linking Number* called *Locking*; here the archi-typical case of the *Borromean*.

Before going further, we leave to the reader to get a better grip on what is at hand by constructing the exercises.

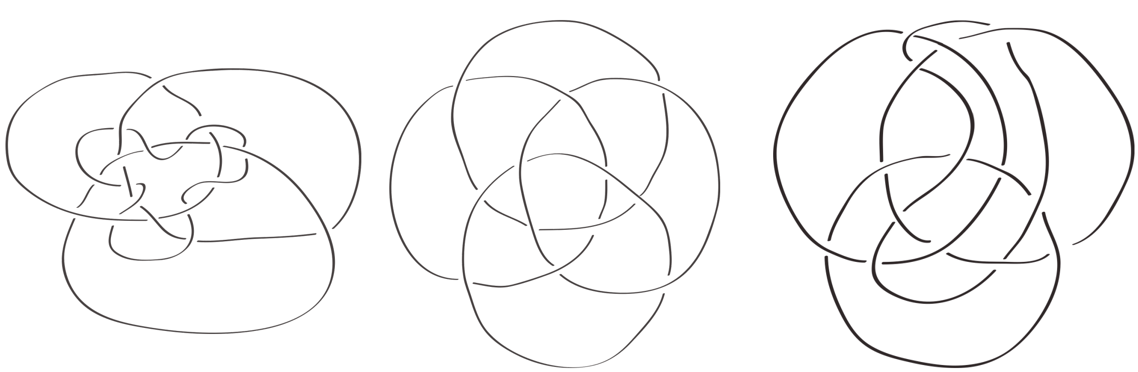
1) Show that the *Reidemeister* moves are invariants of *isotopy*, i.e., that the addition of crossings in a collection of *Reidemeister* moves are always equal to zero no matter how you slide the arcs. For instance, although it is easy to show that *Reidemeister II* is equal to zero:



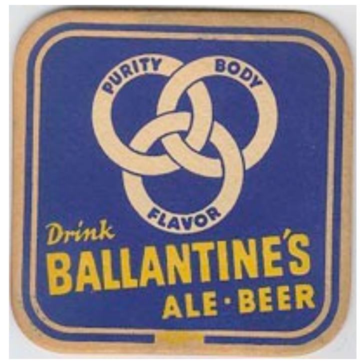
it is less easy to show how *Reidemeister I* and *III* are equal to zero. Explain your response.

2) Undo, if you can, any of the 3-component configurations depicted in fig. X. Just because you cannot undo them does not mean they can not be undone. Use the an argument with the Reidemeister moves to give a combinatorial proof that any configuration with a linking number **L(K) ≥ 1** is *really* linked. Devise an argument to show that if, however, **L(K) = 0**, this does not mean it is a trivial link.

3) Label the crossings of the following configurations and compute the linking number **L(X)**. Are they *linked*, *locked*, or *trivial*?



4) Lacan labeled the Borromean with R.S.I. (Real, Symbolic, Imaginary) which caused several bouts of delirium in France and the journalistic writings on Lacan to explain everything from the Three Muskateers to the Trinity; while it seems in the U.S.A. the only ones to have taken notice was a beer manufacturer:



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